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Threshold Ages for the Relation between Lifetime Entropy and Mortality Risk

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Threshold Ages for the Relation between Lifetime Entropy and Mortality Risk*

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Abstract

We study the effect of a change in age-specific probability of death on risk about the duration of life measured by Shannon's entropy defined to the base 2. We first show that a rise in the probability of death at age n increases lifetime entropy at age $k \leq n$ if and only if the quantity of information revealed by the event of a death at age n exceeds lifetime entropy at age $n+1$ divided by the probability to survive from age k to age $n+1$. There exist, under general conditions, two threshold ages: first, a low threshold age below which a rise in mortality risk decreases lifetime entropy, and above which it raises lifetime entropy; second, a high threshold age above which a rise in mortality risk reduces lifetime entropy. Using French life tables, we show that the gap between those two threshold ages has been increasing over the last two centuries.

Keywords: mortality risk, lifetime entropy, threshold age.

JEL classification codes: J10, D83.

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1 Introduction

Risk about the duration of life is a major dimension of human condition. Whereas all individuals know that they will die one day, no one knows precisely *when* one's death will take place. This risk is faced by everyone, but is somewhat diffuse and abstract, and thus hard to quantity in an intuitive way.[1]

In a recent work, Meyer and Ponthiere (2019) proposed to quantify risk about the duration of life by means of Shannon's entropy index defined to the base 2 (see Shannon 1948). Lifetime entropy is defined as follows:[2]

$$H_k = -\sum_{i=k}^{115} p_{i,k} \log_2 (p_{i,k}) \tag{1}$$

where H_k denotes lifetime entropy at age k and $p_{i,k}$ is the probability of a life of length i for an individual of age k.

Lifetime entropy H_k is the mathematical expectation of the quantity of information revealed by the event of a life of a particular length $i \geq k$, or, alternatively, of the quantity of information revealed by the event of a death at an age $i \geq k$. As such, H_k can be regarded as the informational equivalent of the standard life expectancy. Instead of measuring the mathematical expectation of the duration of life, H_k measures the mathematical expectation of the quantity of information revealed by a particular duration of life.

A major advantage of Shannon's lifetime entropy with respect to other measures of risk about the duration of life, such as Kannisto's coefficient ${}_{10}C_{50}$ (Kannisto 2000), the standard deviation of the age at death (Lan Karen Cheung and Robine 2007, Edwards and Tuljapurkar 2005) and Gini indexes of the length of life (Smits and Monden 2009), is to measure risk about the duration of life in terms of bits, i.e. the quantity of information revealed by tossing a fair coin. As such, that indicator makes the - quite abstract - risk about the duration of life commensurable or comparable with the risk involved in tossing a given number of fair coins, a life experience with which individuals have some familiarity, since at least the Roman Empire.[3]

Whereas the effect of a change of an age-specific probability of death on standard life expectancy is unambiguous, the same is not true as far as its effect on its informational equivalent - i.e. human lifetime entropy - is concerned. Hence an interesting question is to know how sensitive lifetime entropy is to a change in age-specific probability of death. Does a rise in mortality risk at a given age tend to increase, or, on the contrary, to decrease risk about the duration of life measured by lifetime entropy?

The goal of this paper is to examine the impact of a change in age-specific mortality risk on risk about the duration of life, as measured by Shannon's life-

[1] On the various measures of risk about the duration of life, see Wilmoth and Horiuchi (1999) and Van Raalte and Caswell (2013).

[2] That lifetime entropy index is close to the one used in Hill (1993) and Noymer and Coleman (2014), but differs regarding the basis: we use base 2, whereas Hill (1993) uses base e and Noymer and Coleman (2014) use base 10.

[3] See Lanciani (1892).

time entropy index. For that purpose, we first propose to decompose that effect into its various components, to identify the necessary and sufficient condition under which a rise in mortality risk at a given age contributes to increase lifetime entropy. Then, we use that condition to study how the sign of the effect of a change in mortality on lifetime entropy varies with the age.

Anticipating our results, we first show that a rise in the probability of death at age n increases lifetime entropy at age $k \leq n$ if and only if the quantity of information revealed by the event of a death at age n exceeds lifetime entropy at age $n+1$ divided by the probability to survive from age k to age $n+1$. Then, using that condition, we show that there exist, under general conditions, two threshold ages: first, a low threshold age below which a rise in mortality risk decreases lifetime entropy, and above which it raises lifetime entropy; second, a high threshold age above which a rise in mortality risk reduces lifetime entropy. Finally, using French life tables, we identify those two threshold ages, and we show that the gap between those two threshold ages has been increasing over the last two centuries.

Our work is clearly related to Zhang and Vaupel (2009), which examines the effect of averting deaths on life disparity measured by life expectancy lost due to death. Using a framework with age as a continuous variable, Zhang and Vaupel (2009) showed that there exist, under general conditions on Keyfitz's entropy of the life table (Keyfitz 1977), a unique threshold age below which averting deaths reduces life disparity, and above which averting deaths increases life disparity. In particular, Zhang and Vaupel (2009) show that, if Keyfitz's life table entropy is less than 1, such a threshold age exists and is unique, whereas, if Keyfitz's life table entropy is larger than 1, averting deaths at any age increases life disparity, whereas if Keyfitz's life table entropy equals 1, averting deaths at age 0 has no effect, but averting deaths at any higher age increases life disparity.

In comparison to Zhang and Vaupel (2009), our approach differs on three main grounds. First, we focus on the effect of a change in mortality on risk about the duration of life not measured by means of lifetime disparity as measured by life expectancy lost due to death, but by means of Shannon's lifetime entropy index defined to the base 2. Second, at the technical level, our framework is in discrete time rather than in continuous time, which makes the identification of threshold ages more difficult to prove analytically. Third, at the level of results, whereas Zhang and Vaupel (2009) identify a *unique* threshold age using conditions on life table entropy, we identify, on the contrary, not one, but *two* threshold ages, by making assumptions on the pattern of age-specific mortality. The high threshold age that we identify is such that, below that age, a rise in mortality risk raises lifetime entropy, whereas, above that age, a rise in mortality risk reduces lifetime entropy, in a way that is quite similar to the threshold age studied by Zhang and Vaupel (2009). In addition, we show that there exists also another, lower threshold age, below which a rise in mortality reduces lifetime entropy. We show that this low threshold age, which was equal to 6 years in the early 19th century, has turned out to vanish to age 0 in the second half of the 20th century, leaving us with a unique threshold age, below which a rise in mortality raises lifetime entropy, and above which a rise in mortality reduces lifetime entropy.

The rest of this paper is organized as follows. Section 2 derives the necessary and sufficient condition under which a rise in age-specific mortality risk increases lifetime entropy. Then, Section 3 uses that condition to identify two threshold ages, around which the sign of the effect of a rise in age-specific mortality risk increases lifetime entropy changes. Section 4 illustrates our findings numerically, using life tables for France from the Human Mortality Database. Section 5 provides further decompositions of the effects at work, still using French life tables. Concluding remarks are left for Section 6.

2 Relationship

We consider a discrete time model where age is a natural number between the minimum age 0 and the maximum age $M > 0$. The probability of death at age $k \geq 0$, conditionally on survival to that age, is denoted by d_k.

We measure risk about the duration of life by means of Shannon's lifetime entropy index defined to the base 2. Using the identity $p_{i,k} \equiv s_{i,k} d_i$, where $s_{i,k} = \prod_{j=k}^{i-1} (1 - d_j)$ is the probability of survival to age i for an individual of age k, Shannon's lifetime entropy index at age k can be rewritten as:

$$H_k = -\sum_{i=k}^{M} \left(\prod_{j=k}^{i-1} (1 - d_j) d_i \right) \log_2 \left(\prod_{j=k}^{i-1} (1 - d_j) d_i \right) \tag{2}$$

Let us now examine the effect of a variation of the probability of death at age $n \geq k$ on lifetime entropy at age k. Our results are summarized in Proposition 1, which states the necessary and sufficient condition under which a rise of the probability of death at age n increases lifetime entropy at age k. Actually, Proposition 1 states that a rise in the probability of death at age n increases lifetime entropy at age k if and only if the quantity of information revealed by the event of a death at age n (measured by means of Wiener's entropy) exceeds Shannon's lifetime entropy at age $n + 1$ divided by the probability to survive from age k to age $n + 1$.

Proposition 1 *A rise in the probability of death at age n d_n increases (resp. decreases) lifetime entropy at age k if and only if:*

$$W(s_{n,k}) + W(d_n) > \text{ (resp. } < \text{)} \frac{H_{n+1}}{s_{n+1,k}}$$

where $W(x) = -\log_2(x)$ is Wiener's entropy (Wiener 1948), the quantity of information revealed by the occurrence of a single event with probability x.

Proof. Shannon's lifetime entropy at age k can be rewritten in terms of

Wiener's entropy indexes as follows:[4]

$$
\begin{aligned}
H_k &= \sum_{i=k}^{M}\left(\prod_{j=k}^{i-1}(1-d_j)d_i\right)\left[-\log_2\left(\prod_{j=k}^{i-1}(1-d_j)\right)-\log_2\left(d_i\right)\right] \\
&= \sum_{i=k}^{M}\left(\prod_{j=k}^{i-1}(1-d_j)d_i\right)\left[W\left(\prod_{j=k}^{i-1}(1-d_j)\right)+W\left(d_i\right)\right] \qquad (3)
\end{aligned}
$$

where $W(x)=-\log_2(x)$. In order to examine the impact of a variation of the probability of death at age $n \geq k$ on H_k, let us decompose H_k further:

$$
\begin{aligned}
H_k &= \sum_{i=k}^{n-1}\left(\prod_{j=k}^{i-1}(1-d_j)d_i\right)\left[W\left(\prod_{j=k}^{i-1}(1-d_j)\right)+W\left(d_i\right)\right] \\
&\quad+\left(\prod_{j=k}^{n-1}(1-d_j)d_n\right)\left[W\left(\prod_{j=k}^{n-1}(1-d_j)\right)+W\left(d_n\right)\right] \\
&\quad+\sum_{i=n+1}^{M}\left(\prod_{j=k}^{i-1}(1-d_j)d_i\right)\left[W\left(\prod_{j=k}^{i-1}(1-d_j)\right)+W\left(d_i\right)\right] \qquad (4)
\end{aligned}
$$

Using that decomposition, let us now compute the derivative $\frac{\partial H_k}{\partial d_n}$:

$$
\begin{aligned}
\frac{\partial H_k}{\partial d_n} &= \prod_{j=k}^{n-1}(1-d_j)\left[W\left(\prod_{j=k}^{n-1}(1-d_j)\right)+W\left(d_n\right)\right]+\left(\prod_{j=k}^{n-1}(1-d_j)d_n\right)W'\left(d_n\right) \\
&\quad+\sum_{i=n+1}^{M}\left(-\prod_{j=k\backslash n}^{i-1}(1-d_j)d_i\right)\left[W\left(\prod_{j=k}^{i-1}(1-d_j)\right)+W\left(d_i\right)\right] \\
&\quad+\sum_{i=n+1}^{M}\left(\prod_{j=k}^{i-1}(1-d_j)d_i\right)\left[W'\left(\prod_{j=k}^{i-1}(1-d_j)\right)\left(-\prod_{j=k\backslash n}^{i-1}(1-d_j)\right)\right]
\end{aligned}
$$

[4] See Meyer and Ponthiere (2019) on the relation between Shanon's lifetime entropy index and Wiener's entropy of the event "death at age k (conditionally on survival to that age".

where $W'(x) = \frac{-1}{x\ln(2)} < 0$. One can rewrite this as:

$$\begin{aligned}
\frac{\partial H_k}{\partial d_n} = & \underbrace{s_{n,k}\left[W(s_{n,k}) + W(d_n)\right]}_{\text{effect of } \Delta d_n \text{ on likelihood of life of length } n\ (+)} \\
& + \underbrace{s_{n,k} d_n W'(d_n)}_{\text{effect of } \Delta d_n \text{ on information revelation of life of length } n\ (-)} \\
& \underbrace{-\frac{1}{1-d_n} H_{n+1}}_{\text{effect of } \Delta d_n \text{ on likelihood of life of length } > n\ (-)} \\
& + \sum_{i=n+1}^{M} \underbrace{(s_{i,k} d_i)\left[W'(s_{i,k})\left(-\frac{s_{i,k}}{(1-d_n)}\right)\right]}_{\text{effect of } \Delta d_n \text{ on information revelation of life of length } > n\ (+)}
\end{aligned}$$

Note that this expression can be simplified further. Indeed, the second term can be rewritten as:

$$s_{n,k} d_n W'(d_n) = s_{n,k} d_n \frac{-1}{d_n \ln(2)} = s_{n,k} \frac{-1}{\ln(2)}$$

while the fourth term can be rewritten as:

$$\sum_{i=n+1}^{M} (s_{i,k} d_i)\left[W'(s_{i,k})\left(-\frac{s_{i,k}}{(1-d_n)}\right)\right] = \frac{1}{\ln(2)} \frac{1}{(1-d_n)} \underbrace{\sum_{i=n+1}^{M} (s_{i,k} d_i)}_{= s_{n,k}(1-d_n)} = \frac{s_{n,k}}{\ln(2)}$$

since $1 = \sum_{i=k}^{n} s_{i,k} d_i + \sum_{i=n+1}^{M} s_{i,k} d_i \iff \sum_{i=n+1}^{M} s_{i,k} d_i = 1 - \sum_{i=k}^{n} s_{i,k} d_i = s_{n,k}(1-d_n)$.

Thus the second and the fourth terms of $\frac{\partial H_k}{\partial d_n}$ cancel out. We thus have:

$$\begin{aligned}
\frac{\partial H_k}{\partial d_n} \gtrless 0 & \iff s_{n,k}\left[W(s_{n,k}) + W(d_n)\right] - \frac{1}{1-d_n} H_{n+1} \gtrless 0 \\
& \iff W(s_{n,k}) + W(d_n) \gtrless \frac{H_{n+1}}{s_{n+1,k}}
\end{aligned}$$

■

Proposition 1 states that the sign of the effect of a change in mortality risk at age n on lifetime entropy at age k is ambiguous, and depends on whether the quantity of information revealed by the event of a death at age n exceeds Shannon's lifetime entropy at age $n+1$ divided by the probability to survive from age k to age $n+1$. When the quantity of information revealed by the event of a death at age n is higher than Shannon's lifetime entropy at age $n+1$ divided by the probability to survive from age k to age $n+1$, then a rise in the risk of death at age n increases lifetime entropy. Otherwise, when the quantity of information revealed by the event of a death at age n is lower than $\frac{H_{n+1}}{s_{n+1,k}}$, then a rise of mortality risk at age n reduces lifetime entropy.

Whether the condition stated in Proposition 1 is satisfied or not is likely to depend on the particular age n at which mortality varies, since age n affects the quantity of information revealed by the event of a death at age n, i.e. $W(s_{n,k}) + W(d_n)$ on the LHS of the condition, as well as the probability to survive to age $n+1$ on the RHS of the condition. The next section examines in details the relation between the age and the effect of mortality change on lifetime entropy.

3 Existence of threshold ages

The previous section identified a simple condition that determines the sign of the effect of a variation of an age-specific probability of death on lifetime entropy measured by means of Shannon's index defined to the base 2. That condition can be used to show that there exist, under general conditions on the age-pattern of mortality, two threshold ages, around which the sign of the derivative $\frac{\partial H_k}{\partial d_n}$ varies. For that purpose, this section will focus exclusively on the variation of lifetime entropy at birth, that is, on how the age affects the sign of the derivative $\frac{\partial H_0}{\partial d_n}$. Proposition 2 summarizes our results.

Proposition 2 *Assume that the risk of death is first decreasing with the age during childhood, and, then, increasing with the age during the rest of life. There exist two threshold ages a_1 and a_2 with $0 \leq a_1 < a_2$ such that:*

- *when $n < a_1$, a rise in d_n decreases lifetime entropy at birth H_0;*
- *when $a_1 < n < a_2$, a rise in d_n increases lifetime entropy at birth H_0;*
- *when $a_2 < n$, a rise in d_n decreases lifetime entropy at birth H_0.*

Proof. Let us start from the condition of Proposition 1.

$$\frac{\partial H_0}{\partial d_n} \gtrless 0 \iff W(s_{n,0}) + W(d_n) \gtrless \frac{H_{n+1}}{s_{n+1,0}}$$

Let us first prove the existence and uniqueness of the low threshold age $a_1 \geq 0$.

At age 0 ($n = 0$), the condition becomes:

$$\frac{\partial H_0}{\partial d_0} \gtrless 0 \iff W(d_0) \gtrless \frac{H_1}{(1-d_0)}$$

The higher d_0 is, and the lower the LHS is, while the higher the RHS is. We can show that there exists a threshold for infant mortality $\bar{d}_0$ such that for $d_0 > \bar{d}_0$ we have $\frac{\partial H_0}{\partial d_0} < 0$ and for $d_0 < \bar{d}_0$ we have $\frac{\partial H_0}{\partial d_0} > 0$.

Assume that d_0 tends to 0. Then the LHS of the above condition $W(d_0) = -\log_2(d_0)$ tends to $+\infty$, while the RHS tends to H_1, which is finite, thus the

LHS exceeds the RHS, implying that $\frac{\partial H_0}{\partial d_0} > 0$. Assume, on the contrary, that d_0 tends 1. Then the LHS tends to 0, while the RHS tends to $+\infty$. This implies that the RHS exceeds the LHS, implying that $\frac{\partial H_0}{\partial d_0} < 0$.

By continuity, there exists a threshold of infant mortality $\bar{d}_0$ in $]0,1[$ such that $W\left(\bar{d}_0\right) = \frac{H_1}{(1-\bar{d}_0)}$, at which a marginal change in infant mortality has no impact on lifetime entropy, i.e. $\frac{\partial H_0}{\partial d_0} = 0$. That threshold is unique, since the LHS is strictly decreasing in d_0, while the RHS is strictly increasing in d_0.

Hence, when infant mortality is higher than the threshold (which depends on H_1), a rise in infant mortality reduces lifetime entropy. When it is lower than the threshold, it raises lifetime entropy.

Let us now move away from the first life-year. With the age, mortality falls during infancy and then rises during adulthood. Consider now young adult mortality, at which d_n is very low and increasing with the age. In general $W\left(s_{n,0}\right)$ tends to 0 (since $s_{n,k}$ is close to 1) and $s_{n+1,k}$ tends also to 1, so that the condition becomes:

$$\frac{\partial H_0}{\partial d_n} \gtrless 0 \iff W\left(d_n\right) \gtrless H_{n+1}$$

Substituting for H_{n+1}, that expression becomes:

$$\frac{\partial H_0}{\partial d_n} \gtrless 0 \iff W\left(d_n\right) \gtrless H_n + \frac{d_n}{1-d_n}\left[H_n - W(d_{n)}\right] - (1-d_n)W(1-d_n)$$

Since d_n is very low during young adulthood, the condition can be approximated by:

$$\frac{\partial H_0}{\partial d_n} \gtrless 0 \iff W\left(d_n\right) \gtrless H_n + 0 - 0$$

When mortality d_n is low and increasing with the age, we know from Meyer and Ponthiere (2019) that the amount of information revealed by the event of a death at an age n, i.e. $W(d_n)$, is higher than the mathematical expectation of the amount of information revealed by the event of a death at an age $i \geq n$, i.e. H_n, so that the LHS exceeds the RHS, implying that $\frac{\partial H_k}{\partial d_n} > 0$. Thus, when considering young adults, the effect of a change of mortality risk on lifetime entropy is positive.

Combining this with what we know concerning infancy, we see that two cases can arise concerning the low threshold a_1:

- Either infant mortality is higher than $\bar{d}_0$, so that we first have that a higher mortality reduces entropy during early childhood, and then raises entropy during young adulthood, in which case there must exist a threshold age $a_1 > 0$ below which $\frac{\partial H_0}{\partial d_0} < 0$ and above which $\frac{\partial H_0}{\partial d_0} > 0$;

- Or infant mortality is lower than $\bar{d}_0$, so that it is the case that, during childhood and young adulthood, we have $\frac{\partial H_0}{\partial d_0} > 0$, in which case the threshold age equals 0, and we have $a_1 = 0$.

Consider now the existence of second threshold age a_2.

We know from above that, during young adulthood, we have $\frac{\partial H_k}{\partial d_n} > 0$. If the mortality risk is, during adulthood, increasing monotonically with the age, the impact of a rise of the risk of death on lifetime entropy at birth becomes smaller and smaller as the mortality risk goes up. To see this, rewrite the condition of Proposition 1 as:

$$\frac{\partial H_k}{\partial d_n} \gtrless 0 \iff W(d_n) \gtrless \frac{H_{n+1}}{s_{n+1,k}} - W(s_{n,k})$$

Take now the second-order derivative with respect to d_n. Since $s_{n,k}$ and H_{n+1} do not depend on d_n, we have:

$$\frac{\partial^2 H_k}{\partial d_n^2} = \underbrace{W'(d_n)}_{-} + \underbrace{\frac{H_{n+1}(-s_{n,k})}{(s_{n+1,k})^2}}_{-} < 0$$

Thus the impact of d_n on lifetime entropy is positive but decreasing. Thus, as mortality risk rises with the age (beyond age 20), the rise in lifetime entropy becomes smaller and smaller.

Does there exist a level of d_n such that $\frac{\partial H_k}{\partial d_n} = 0$ and then turns to be negative? The answer is positive. To see this, take very high ages. As n tends to be very large, we have:

$$\frac{\partial H_0}{\partial d_n} = W(s_{n,0}) + W(d_n) - \frac{H_{n+1}}{s_{n+1,0}} = \frac{s_{n+1,0}W(s_{n,0}) - H_{n+1}}{s_{n+1,0}} + W(d_n)$$

When n is very large, we have that $s_{n,0}$ and $s_{n+1,0}$ tend to 0, so that the above expression can be approximated by:

$$\frac{\partial H_0}{\partial d_n} = \frac{\sim 0 - H_{n+1}}{\sim 0} + W(d_n) < 0$$

Thus, at high ages, we have that $\frac{\partial H_0}{\partial d_n} < 0$.Thus, since we have $\frac{\partial H_0}{\partial d_n} > 0$ during young adulthood, there must exist a second threshold age, a_2, below which $\frac{\partial H_0}{\partial d_n} > 0$ and above which $\frac{\partial H_0}{\partial d_n} < 0$.

This completes the proof of the existence of a threshold a_2. Note that the uniqueness of that second threshold age is guaranteed by the monotonicity of mortality with age for adults.

Finally, note that, when n is very large, and tends to the maximum age M, we have $H_{n+1} \to 0$, so that, using Hospital Rule, we have, from the above condition, that: $\frac{\partial H_0}{\partial d_n}$ tends towards 0. Thus, although the impact of a rise of d_n on H_0 is negative when $n > a_2$, this negative effect tends to vanish to zero when the age n approaches the maximum age M. ■

Proposition 2 states that, under general conditions on the age-mortality pattern, the effect of a variation of the probability of death d_n on lifetime entropy varies with the age n. Proposition 2 states that there exists two threshold ages: first, a low threshold age a_1, below which a rise of d_n decreases lifetime entropy, and above which a rise of d_n increases lifetime entropy, at least until a second threshold age a_2 is reached, beyond which a rise of d_n decreases lifetime entropy. Thus, in the light of Proposition 2, it appears that how a variation of the probability of death affects lifetime entropy varies significantly with the age: for very low ages and very high ages, a rise of d_n decreases lifetime entropy, whereas for intermediate ages the opposite holds and a rise of d_n increases lifetime entropy.

When interpreting Proposition 2, one should be cautious about the level of the low threshold age a_1. Actually, as discussed in the proof, it is possible that, when infant mortality is sufficiently low, the first threshold age equals 0, so that, for all ages below a_2, a rise of d_n increases lifetime entropy.

4 Numerical illustration

In order to illustrate the existence of the two threshold ages around which the relation between mortality risk and lifetime entropy varies, this section uses life tables for France from the Human Mortality Database. For the simplicity of presentation, this section will use, for women and men, five period life tables, for years 1816, 1900, 1950, 1980 and 2016.

Figure 1 shows, for each of those life tables, the impact of a variation of the probability of death d_n on lifetime entropy at birth H_0 as a function of the age n (x axis). Figure 1 focuses on French women, but a quite similar picture prevails also for men (see in the Appendix).

Several observations can be made, which allow us to illustrate the results obtained in the previous sections.

A first important observation is that, based on the survival conditions prevailing in 1816, 1900 and 1950, there exists a strictly positive low threshold age a_1 below which a rise of d_n reduces lifetime entropy at birth. However, when considering the life tables for 1980 and 2016, that low threshold age has vanished to 0. That result is due to the strong fall of infant mortality in the second part of the 20th century.

A second observation concerns the existence of a high threshold age a_2, beyond which a rise of the probability of death reduces lifetime entropy at birth. That second threshold age is shown to have moved significantly to the right, i.e. to higher values, when shifting from the 1816 life table to the 2016 life table.

Figure 1 thus shows that, during the major part of life, a higher probability of death at a given age tends to increase lifetime entropy at birth. However, the size of that age interval has tended to vary over time. That age interval was much shorter when considering life tables of 1816 and 1900, because, at those times, there was a low age interval, during childhood, in which a rise of d_n implied a fall of lifetime entropy, and, also, because, at those times, the second threshold age was much lower, and around age 60. Under contemporary survival conditions, the first threshold age has vanished to 0 and the second threshold age is much higher, at about 80 years.

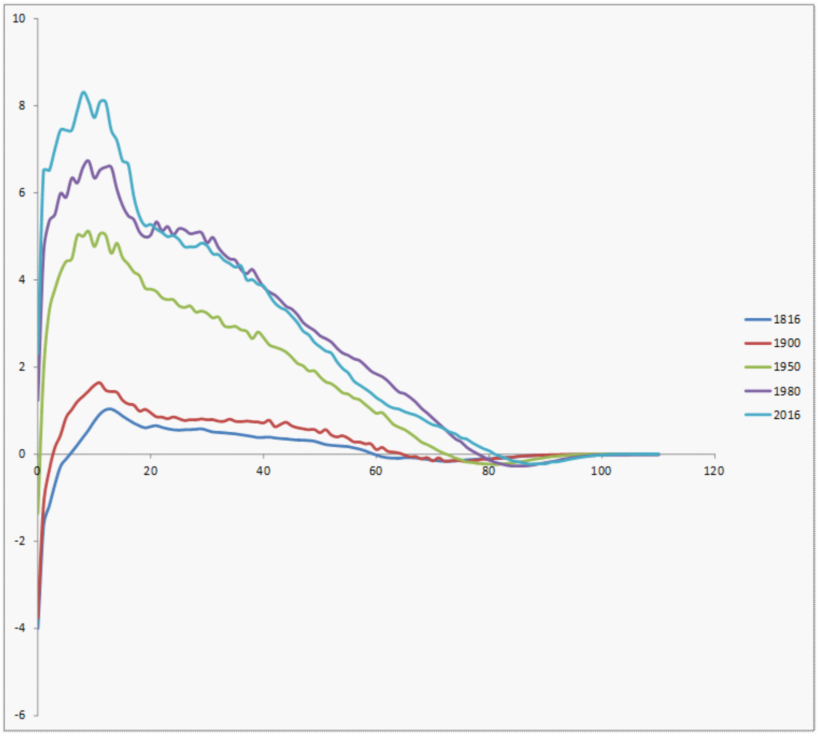

Figure 1: Effect of a variation of d_n on H_0 as a function of the age n, French women.

In order to have a more accurate view of those changes, Table 1 summarizes the levels of the two threshold ages a_1 and a_2 for men and women, under the five life tables under comparison. That table shows that the low threshold age has tended to decrease over time, and to vanish to zero. This change is due to the fall of infant mortality. As such, Table 1 illustrates well the above discussions in the proof of Proposition 2. Table 1 also shows that the second threshold age has tended to increase over time.

years	women		men	
	a_1	a_2	a_1	a_2
1816	6 years	60 years	6 years	60 years
1900	3 years	65 years	3 years	60 and 62 years
1950	1 year	73 years	1 year	68 years
1980	0 year	79 years	0 year	70 years
2016	0 year	81 years	0 year	81 years

Table 1: Threshold ages for women and men, France.

Another interesting observation concerning Table 1 is the lengthening, over time, of the age interval during which a rise of the probability of death tends to increase lifetime entropy. That age interval can be measured as the difference between the two threshold ages, i.e. $a_2 - a_1$. It was equal, for women, to 54 years in 1816, and is as large as 81 years for 2016. Thus the part of life during which a rise of mortality increases lifetime entropy is now much larger than it used to be in the past.

When considering Table 1, a quite surprising result consists of the existence of two high threshold ages for men in 1900. That result is due to the non-monotonicity of the probability of death d_n around ages 60-61 for French men in 1900. That non-monotonicity explains the non-uniqueness of the high threshold age. Note that this does not infirm the validity of Proposition 2, since Proposition 2 assumed the monotonicity of the mortality age-pattern during adulthood, which was a sufficient condition for the uniqueness of the high threshold age a_2. The 1900 life table does not satisfy that monotonicity condition for adult mortality.

5 Further decompositions

In order to better understand the forces at work behind the dynamics of threshold ages a_1 and a_2 over time, Figures 2a and 2b show, for respectively 1816 and 2016, the decomposition of the net effect into its two components, whose levels are here presented in absolute values. The first component (LHS of the condition in Proposition 1) is $W(s_{n,0}) + W(d_n)$, that is, the quantity of information revealed by the event of a death at age n, whereas the second component (RHS of the condition in Proposition 1) is $\frac{H_{n+1}}{s_{n+1,0}}$, that is, lifetime entropy at age $n+1$ divided by the probability to survive to that age.

Concerning the life table 1816, Figure 2a shows that, at low ages of life, mortality was high, so that the quantity of information revealed by the event of a death at age n is lower than the lifetime entropy at age $n+1$ divided by the probability to survive to that age, which explains why the net effect of a variation of d_n is negative at those low ages of life. Then, as the age goes up, mortality falls, which leads to a rise of quantity of information revealed by the event of a death at age n, which equalizes $\frac{H_{n+1}}{s_{n+1,0}}$ at the first threshold age a_1. Then, during about 50 years, the first component dominates the second one, explaining that the rise of d_n raises lifetime entropy at birth. The two curves cross once again around age 60, which is the threshold age a_2. Above that age, the second component $\frac{H_{n+1}}{s_{n+1,0}}$ exceeds the first one, i.e. $W(s_{n,0}) + W(d_n)$, so that a rise of d_n has, at those high ages, the effect of decreasing lifetime entropy.

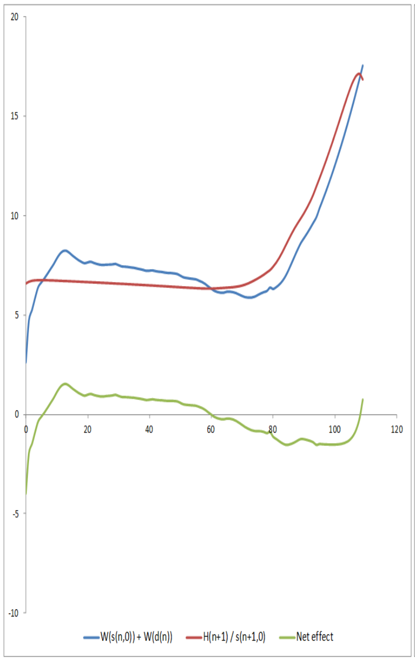

Figure 2a: Decomposition of the effect of d_n on H_0 into the two components, French women, 1816.

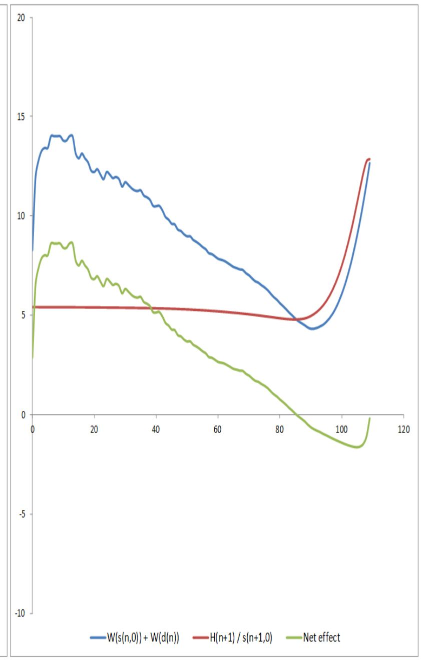

Figure 2b: Decomposition of the effect of d_n on H_0 into the two components, French women, 2016.

As shown on Figure 2b, the overall picture is very different under the 2016 life table. First, infant mortality is much lower, so that, even when considering very young ages of life, the quantity of information revealed by the event of a death at age n, $W(s_{n,0}) + W(d_n)$, is larger than lifetime entropy at age $n+1$ divided by the probability to survive to that age. As a consequence, the low threshold age a_1 equals zero, and the effect of a rise of d_n is, since birth, to increase lifetime entropy, unlike what used to be the case in the early 19th century (Figure 2a). Figure 2b also shows that the effect of d_n on H_0 tends, during adulthood, to decrease with the age, until the second threshold age is reached, and this pattern is due to the decreasing gap between the two components, $W(s_{n,0}) + W(d_n)$ and $\frac{H_{n+1}}{s_{n+1,0}}$. As age goes up, the quantity of information revealed by the event of a death goes down, and at the same time the decrease of the survival probability $s_{n+1,0}$ pushes the second component up. The two curves cross at the second threshold age a_2, beyond which a rise of the probability of death will tend to reduce rather than increase lifetime entropy.

6 Concluding remarks

The relation between the probability of death at a given age and lifetime entropy is not trivial: depending on the age, a rise of the probability of death may either increase or decrease lifetime entropy, leading to a larger or a smaller risk about the duration of life.

In order to examine that relationship, this paper first derived a necessary and sufficient condition under which a rise of the probability of death at a given age increases lifetime entropy measured by Shannon's entropy index defined to the base 2. That condition is actually quite intuitive: a rise in the probability of death at age n increases lifetime entropy at age $k \leq n$ if and only if the quantity of information revealed by the event of a death at age n, measured, by means of Wiener's entropy, by $W(s_{n,k}) + W(d_n)$, exceeds lifetime entropy at age $n+1$ divided by the probability to survive from age k to age $n+1$.

Then, in a second stage, we used that condition to show the existence of two threshold ages a_1 and a_2. When the age is below a_1, a rise of the risk of death decreases lifetime entropy at birth, whereas when the age lies between a_1 and a_2, it increases lifetime entropy at birth. Then, for ages exceeding a_2, a rise in the probability of death reduces lifetime entropy.

Finally, using French life tables over the last two centuries, we showed that the first threshold age a_1 has tended to decrease over time, with the decrease of infant mortality. Whereas it used to be about 6 years in 1816, it is as low as 0 year in 1980 and after. On the contrary, the threshold age a_2 has tended to increase in the last two centuries, from about 60 years in 1816 to 81 years in 2016. As a consequence of those two patterns, the age interval between a_1 and a_2, in which a rise in d_n causes a rise of H_0, has been widening over time, from about 54 years in 1816 to 81 years in 2016. Hence, in comparison to more distant epochs, an increasingly larger period of life is characterized by a positive relation between the risk of death and lifetime entropy.

All in all, this study shows that the relation between the risk of death and lifetime entropy varies with the age, and that this relation has also changed over the last two centuries. The existence of a *unique* strictly positive threshold age, as studied in Zhang and Vaupel (2009), dates back to the second part of the 20th century. Before that, there used to be another, lower, threshold age, below which a rise of mortality risk leads to a fall of lifetime entropy.

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8 Appendix

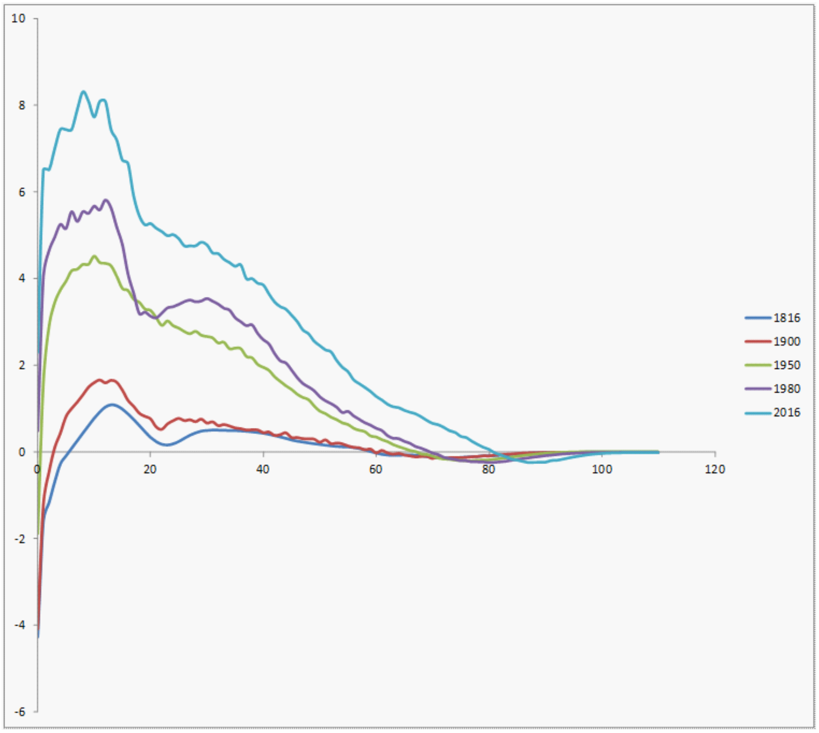

Figure A1: Effect of a variation of d_n on H_0 as a function of the age n, French men.